

Finite Scale Lyapunov Analysis of Temperature Fluctuations in Homogeneous Isotropic Turbulence

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Abstract

The study analyzes the temperature fluctuations in incompressible homogeneous isotropic turbulence through the finite scale Lyapunov analysis of the relative motion between two fluid particles. The analysis determines the temperature fluctuations through the Lyapunov theory of the local deformation, using the thermal energy equation. The study provides an explanation of the mechanism of temperature cascade, leads to the closure of the Corrsin equation, and describes the statistics of the longitudinal temperature derivative. The results here obtained show that, in the case of self-similarity of velocity and temperature correlations, the temperature spectrum exhibits the scaling laws κ^n , with $n \approx -5/3$, -1 and $-17/3 \div -11/3$ depending upon the flow regime, and in agreement with the theoretical arguments of Obukhov–Corrsin and Batchelor and with the numerical simulations and experiments known from the literature. The longitudinal temperature derivative PDF is found to be a non-gaussian distribution function with null skewness, whose intermittency rises with the Taylor scale Péclet number.

This study applies also to any passive scalar which exhibits diffusivity.

Keywords: Lyapunov Analysis, Corrsin equation, von Kármán-Howarth equation, Self-Similarity

1. Introduction

This work proposes the adoption of the finite-scale Lyapunov theory, for studying the temperature fluctuations in incompressible homogeneous isotropic turbulence in an infinite domain. The study is mainly motivated by the fact that, in isotropic turbulence, the temperature spectrum $\Theta(\kappa)$

exhibits several scaling laws κ^n in the different wavelength ranges, depending on Taylor scale Reynolds number R and Prandtl number Pr (Corrsin (JAP 1951), Obukhov (1949), Batchelor (1959), Batchelor et al (1959)). This is due to the combined effect of these parameters which produces a peculiar connection between temperature fluctuations, fluid deformation and velocity field.

For large values of R and Pr , Corrsin (JAP 1951) and Obukhov (1949) argued, through the dimensional analysis, that $\Theta(\kappa) \approx \kappa^{-5/3}$ in the so-called inertial-convective subrange (see Fig. 1). Batchelor (1959) considered the isotropic turbulence at high Prandtl number, when R is assigned. There, the author assumed that, at distances less than the Kolmogorov scale, the temperature fluctuations are mainly related to the strain rate associated to the smallest scales of the velocity field. As the result, he showed that $\Theta \approx \kappa^{-1}$ in the so-called viscous-convective interval, a region where the scales are less than the Kolmogorov length (see Fig. 1). Different experiments dealing with the grid turbulence (Gibson & Schwarz (1963), Mydlarski & Warhaft (1998)) and calculations of the temperature spectrum through numerical simulations (Donzis et al (2010) and references therein) indicate that $\Theta(\kappa)$ follows the previous scaling laws.

On the contrary, when Pr is very small, the high fluid conductivity determines quite different situations with respect to the previous ones. Batchelor et al (1959) analyzed the small-scale variations of temperature fluctuations in the case of large conductivity, and found that $\Theta(\kappa) \approx \kappa^{-17/3}$, whereas Rogallo et al (1989) determined the temperature spectrum through numerical simulations of a passive scalar convected by a velocity field with zero correlation time. Rogallo et al (1989) showed that, when the kinetic energy spectrum follows the Kolmogorov law $E(\kappa) \approx \kappa^{-5/3}$, the temperature spectrum varies according to $\Theta(\kappa) \approx \kappa^n$, with $n \approx -11/3$.

Furthermore, experiments of grid turbulence state that both temperature and velocity correlations are linked each other when $Pr = O(1)$ and that decay rate and characteristic scales depend on the initial conditions. Specifically, Mills et al (1958) obtained very important data about the air turbulence behind a heated grid. They carried out several measurements of nearly isotropic fluctuations of velocity and temperature at different distances from the grid, and recognized that the temperature correlation f_θ is roughly equal to the longitudinal velocity correlation f , and that the triple correlation temperature-velocity p_* is of the order of the triple velocity correlation k . Later, Warhaft & Lumley (1978) experimentally showed that

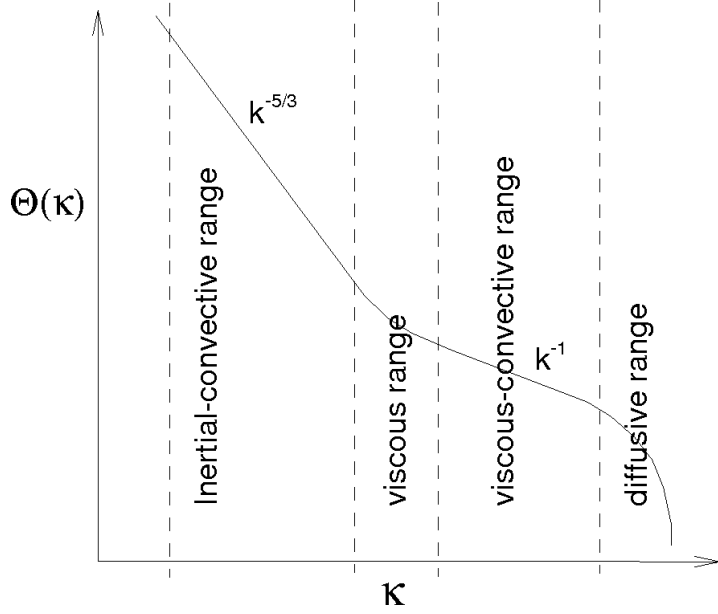


Figure 1: Scheme of the subranges of the temperature spectrum at high Prandtl numbers

spectrum shape and decay rate depend upon the initial conditions and that the mechanical–thermal time scale ratio tends to a value close to unity.

Another important characteristics of $\Theta(\kappa)$ is the self-similarity. This is related to the idea that the combined effect of temperature and kinetic energy cascade in conjunction with conductivity and viscosity, makes the temperature correlation similar in the time. This property was theoretically studied by George (see George (1988), George (1992) and references therein) which showed that the decaying isotropic turbulence reaches the self-similarity, where $\Theta(\kappa)$ is scaled by the Taylor microscale whose current value depends on the initial condition. Recently, Antonia et al (2004) studied the temperature structure functions in decaying homogeneous isotropic turbulence and found that the standard deviation of the temperature, like the turbulent kinetic energy, follows approximately the similarity over a wide range of length scales. There, the authors used this approximate similarity to calculate the third-order correlations and found satisfactory agreement between measured and calculated functions.

From a theoretical point of view, the properties of $\Theta(\kappa)$ can be investigated through the evolution equation of the temperature spectrum. $\Theta(\kappa)$

is the Fourier-Transform of f_θ which in turn satisfies the Corrsin equation (Corrsin (JAS 1951)). This latter includes the term G , a quantity providing the temperature cascade, directly related to the triple correlation p_* , and responsible for the thermal energy distribution at the several wavelengths. As G depends also on the velocity fluctuations, the Corrsin equation requires the knowledge of the velocity correlation f , thus it must be solved together to the von Kármán–Howarth equation. On this argument, some work has been written. For instance, Baev & Chernykh (2010) (and references therein) studied temperature and kinetic energy spectra adopting a closure model for the Corrsin and von Kármán–Howarth equations based on the gradient hypothesis, which incorporates empirical constants. Nevertheless, to the author knowledge, the estimation of $\Theta(\kappa)$ based on the theoretical analysis of the closure of von Kármán–Howarth and Corrsin equations has not received due attention.

This is the motivation of the present work, whose main objective is to propose the closure of the Corrsin equation and a description of the statistics of the temperature derivative. The present study is based on the finite-scale Lyapunov theory, just used by de Divitiis (2010) and de Divitiis (2011) for determining the closure of the von Kármán–Howarth equation and the statistics of the velocity difference. Here, this theory gives G in function of f and $\partial f_\theta / \partial r$, and describes also the statistics of the temperature gradient through the Lyapunov analysis of the local strain and the canonical decomposition of temperature and velocity in terms of proper stochastic variables. The closure of the Corrsin equation is obtained considering that G is frame invariant, thus G is calculated in the finite scale Lyapunov basis. The adoption of this basis is revealed to be an usefull choice for determining the analytical expression of G . For what concerns the von Kármán–Howarth equation, the analytical closure proposed by de Divitiis (2010) is here adopted. From the system of equations of von Kármán–Howarth and Corrsin, an ordinary differential system is determined, through the hypothesis of self-similarity for f and f_θ . This differential system is first reduced to a Cauchy’s initial condition problem, then it is numerically solved for several values of R and Pr . The results show that the temperature spectrum exhibits scaling laws whose exponents depends on R and Pr , in agreement with the experimental and theoretical data of the literature (Corrsin (JAP 1951), Obukhov (1949), Rogallo et al (1989), Mills et al (1958), Gibson & Schwarz (1963)). As far as the statistics of the longitudinal temperature gradient is concerned, it is represented by non-gaussian PDF with null skewness and a Kurtosis greater

than three whose value rises with the Péclet number $Pr R$.

2. Temperature correlation equation

For sake of convenience, the procedure to obtain the Corrsin equation is here renewed.

The isotropic homogeneous velocity and temperature fields are considered, where fluid viscosity ν and thermal conductivity k are assigned quantities. The equations of the temperature fluctuation ϑ in two points $\mathbf{x} \equiv (x, y, z)$ and $\mathbf{x}' \equiv \mathbf{x} + \mathbf{r}$, are

$$\frac{\partial \vartheta}{\partial t} + \frac{\partial \vartheta}{\partial x_k} u_k - \chi \frac{\partial^2 \vartheta}{\partial x_k \partial x_k} = 0 \quad (1)$$

$$\frac{\partial \vartheta'}{\partial t} + \frac{\partial \vartheta'}{\partial x'_k} u'_k - \chi \frac{\partial^2 \vartheta'}{\partial x'_k \partial x'_k} = 0 \quad (2)$$

being $\mathbf{r} = (r_x, r_y, r_z)$ the separation vector, $\chi = k/(\rho C_p)$ is the fluid thermal diffusivity, and C_p is the specific heat at constant pressure. In case of homogeneous temperature fluctuations, the temperature correlation is defined as

$$f_\theta = \frac{\langle \vartheta \vartheta' \rangle}{\theta^2} \quad (3)$$

where $\theta = \sqrt{\langle \vartheta^2 \rangle}$ is the standard deviation of the temperature fluctuations, constant in the space.

As well known, the evolution equation of f_θ is determined multiplying Eq. (1) and (2) by ϑ' and ϑ , respectively, and summing the equations (Corrsin (JAS 1951)). The so obtained equation, averaged with respect to the ensemble of the temperature fluctuations, leads to

$$\theta^2 \frac{\partial f_\theta}{\partial t} + f_\theta \frac{d\theta^2}{dt} + \frac{\partial}{\partial r_k} \langle \vartheta \vartheta' (u'_k - u_k) \rangle - 2\chi \theta^2 \left(\frac{\partial^2 f_\theta}{\partial r^2} + \frac{2}{r} \frac{\partial f_\theta}{\partial r} \right) = 0 \quad (4)$$

The first two terms of Eq. (4) express the time variations of f_θ and θ , the third one, arising from the convective terms, provides the mechanism of temperature cascade, whereas the last one, the laplacian in the spherical coordinates of f_θ , describes effects of the thermal diffusion. Because of isotropy

of temperature and velocity fluctuations, the third term can be expressed through the scalar $G(r)$, an even function of $|\mathbf{r}|$

$$\frac{\partial}{\partial r_k} \langle \vartheta \vartheta' (u'_k - u_k) \rangle = -G \quad (5)$$

According to Corrsin (JAS 1951), Corrsin (JAP 1951), G is

$$G = 2 \left(\frac{\partial p_*}{\partial r} + 2 \frac{p_*}{r} \right) \equiv \frac{2}{r^2} \frac{\partial}{\partial r} (p_* r^2) \quad (6)$$

being $p_*(r)$ the triple correlation between temperature fluctuations in \mathbf{x} and \mathbf{x}' , and the velocity component in \mathbf{x} along the direction \mathbf{r}

$$p_*(r) = \frac{\langle u_r \vartheta \vartheta' \rangle}{\theta^2 u} \quad (7)$$

and u_r is the longitudinal component of the velocity fluctuation.

Now, the temperature cascade, responsible for the distribution of thermal energy at the different wave-lengths, does not modify θ , hence $G(0) = 0$ and $G \approx r^2$ near the origin. Therefore, $p_* \approx r^3$ when $r \rightarrow 0$, and for $r = 0$ Eq. (4) gives the evolution equation for θ (Corrsin (JAP 1951))

$$\frac{d\theta^2}{dt} = -12\chi \frac{\theta^2}{\lambda_\theta^2} \quad (8)$$

λ_θ is the scale of the temperature correlation, or Corrsin microscale, defined as (Corrsin (JAS 1951))

$$\lambda_\theta = \sqrt{-\frac{2}{f_\theta''(0)}} \quad (9)$$

where the superscript apex denotes the differentiation with respect to r . As the consequence, the evolution equation of f_θ is

$$\frac{\partial f_\theta}{\partial t} - 12 \frac{\chi}{\lambda_\theta^2} f_\theta - G - 2\chi \left(\frac{\partial^2 f_\theta}{\partial r^2} + \frac{2}{r} \frac{\partial f_\theta}{\partial r} \right) = 0 \quad (10)$$

whose boundary conditions are

$$\begin{aligned} f_\theta(0) &= 1, \\ \lim_{r \rightarrow \infty} f_\theta(r) &= 0 \end{aligned} \quad (11)$$

Note that Eq. (10) depends also on the velocity fluctuations through $G(r)$ whose analytical expression is not given at this stage of this analysis. Therefore, Eq. (10) is not closed and provides only a link between G and f_θ . Accordingly f_θ is related to f . As k and ν are both constant, the von Kármán–Howarth equation (see Eq. (90), Appendix) is independent from f_θ and θ , hence Eq. (90) can be first solved separately, whereas Eq. (10) requires the knowledge of f by means of G .

3. Lyapunov Analysis of the temperature cascade

The purpose of this section is to analyse the flow of the thermal energy cascade with the finite-scale Lyapunov theory used by de Divitiis (2010), and to propose an analytical expression for G which provides the closure of the Corrsin equation. To this end, consider now the expression of G (Eq. (5))

$$G(r) = -\frac{\partial}{\partial r_k} \langle \vartheta \vartheta' (u'_k - u_k) \rangle \quad (12)$$

This is frame invariant, therefore, for sake of convenience, G is expressed in the finite scale Lyapunov basis E_λ . This basis is associated to the problem of the relative motion between two fluid particles (de Divitiis (2010))

$$\begin{aligned} \frac{d\boldsymbol{\rho}}{dt} &= \mathbf{u}(\mathbf{x} + \boldsymbol{\rho}, t) - \mathbf{u}(\mathbf{x}, t), \\ \frac{d\mathbf{x}}{dt} &= \mathbf{u}(\mathbf{x}, t) \end{aligned} \quad (13)$$

where $\boldsymbol{\rho}$ gives the relative position between the particles, and \mathbf{u} varies according to the Navier–Stokes equations. E_λ is defined by means of the solutions $\boldsymbol{\rho}_1$, $\boldsymbol{\rho}_2$ and $\boldsymbol{\rho}_3$ of Eq. (13), whose initial conditions $\boldsymbol{\rho}_1(0)$, $\boldsymbol{\rho}_2(0)$ and $\boldsymbol{\rho}_3(0)$ are mutually orthogonal vectors which satisfy $|\boldsymbol{\rho}_1(0)| = |\boldsymbol{\rho}_2(0)| = |\boldsymbol{\rho}_3(0)| \equiv r$. Specifically, E_λ is obtained through the Gram-Schmidt orthonormalization process applied to $\boldsymbol{\rho}_1(t)$, $\boldsymbol{\rho}_2(t)$ and $\boldsymbol{\rho}_3(t)$.

In E_λ , the velocity difference fluctuation is expressed as (de Divitiis (2010))

$$\Delta \mathbf{u} \equiv \mathbf{u}' - \mathbf{u} = \lambda(r) \mathbf{r} + \boldsymbol{\omega}_\lambda \times \mathbf{r} + \boldsymbol{\zeta} \quad (14)$$

where λ is the maximal finite scale Lyapunov exponent (associated to the length r), defined by $\lambda(r) \approx 1/T \int_0^T d\mathbf{r}/dt \cdot \mathbf{r}/r^2 dt$, and calculated in function

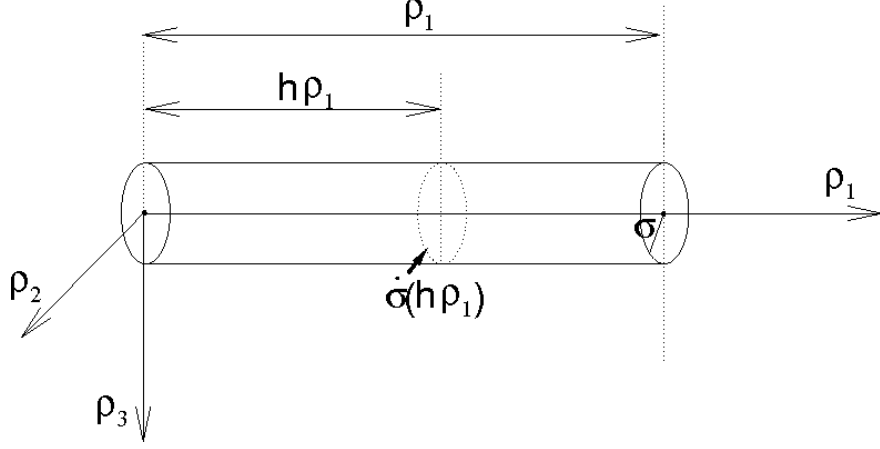


Figure 2: Scheme of Finite-scale Lyapunov basis embedded into a material cylinder, at a given time

of f as (de Divitiis (2010))

$$\lambda(r) = \frac{u}{r} \sqrt{2(1-f)} \quad (15)$$

ω_λ is the angular velocity of E_λ with respect to the inertial frame of reference \mathfrak{R} , and $\zeta \equiv (\zeta_1, \zeta_2, \zeta_3)$, related to the other two exponents, makes $\Delta \mathbf{u}$ a solenoidal field, and is expressed in E_λ as (de Divitiis (2010))

$$\zeta_1 = (\lambda_1 - \lambda)\varrho_1, \quad \zeta_2 = (\lambda_2 - \lambda)\varrho_2, \quad \zeta_3 = (\lambda_3 - \lambda)\varrho_3 \quad (16)$$

where λ_1 , λ_2 and λ_3 are the Lyapunov exponents associated to the directions ϱ_1 , ϱ_2 and ϱ_3 , respectively. As the consequence, λ is a deterministic quantity, whereas ω_λ is a fluctuating variable related to the relative motion between E_λ and \mathfrak{R} . Without lack of generality, the coordinate ϱ_1 is supposed to be associated to the maximal exponent, then $\lambda_1 \rightarrow \lambda$, $\lambda_2 = \lambda_3 \equiv \lambda_\zeta$, ϱ_1 diverges being $|\varrho_1| \gg \gg |\varrho_2|, |\varrho_3|$, thus $\zeta_1 \rightarrow \hat{\zeta}_1 \neq 0$

$$\zeta_1 = \hat{\zeta}_1, \quad \zeta_2 = (\lambda_\zeta - \lambda)\varrho_2, \quad \zeta_3 = (\lambda_\zeta - \lambda)\varrho_3 \quad (17)$$

The exponents $\lambda_2 = \lambda_3 \equiv \lambda_\zeta$ are determined with the continuity equation. With reference to Fig. 2, the equation is written considering, at a given time, the mass balance associated to a material circular cylinder whose axis is parallel to the direction ϱ_1

$$2\frac{\dot{\sigma}}{\sigma} + \frac{\dot{\varrho}_1}{\varrho_1} = 0 \quad (18)$$

where the dot denotes the differentiation with respect to t , $\dot{\sigma}$ is evaluated at the coordinate $h \varrho_1$, with $h \in (0, 1)$, whereas ϱ_1 and $\sigma \equiv \sqrt{\varrho_2^2 + \varrho_3^2}$ are length and diameter of the cylinder. Therefore, $\dot{\varrho}_1/\varrho_1$ and $\dot{\sigma}/\sigma$ identify λ and λ_ζ , respectively, being

$$\lambda_\zeta = -\frac{\lambda}{2} \quad (19)$$

Substituting Eq. (14) into Eq. (12), and taking into account that λ is constant with respect to the operation of statistical average (i.e. $\langle \lambda \rangle = \lambda \langle \cdot \rangle$), G is the sum of three addends

$$G = G_1 + G_2 + G_3 \quad (20)$$

where

$$\begin{aligned} G_1 &= -\frac{\partial}{\partial \varrho_k} (\langle \vartheta \vartheta' \rangle \lambda \varrho_k), \\ G_2 &= -\frac{\partial}{\partial \varrho_k} (\langle \vartheta \vartheta' \rangle \zeta_k), \\ G_3 &= -\frac{\partial}{\partial \varrho_k} (\varepsilon_{kij} \langle \vartheta \vartheta' \omega_i \rangle \varrho_j) \end{aligned} \quad (21)$$

being ε_{kij} the Levi-Civita tensor. G_1 reads as

$$G_1 = -\theta^2 \left(\frac{\partial f_\theta}{\partial r} \lambda r + \frac{f_\theta}{r^2} \frac{\partial}{\partial r} (r^3 \lambda) \right) \quad (22)$$

and G_2 is written taking into account that $\lambda(r) \mathbf{r} + \boldsymbol{\omega}_\lambda \times \mathbf{r} + \boldsymbol{\zeta}$ is solenoidal

$$G_2 = -\frac{\partial \langle \vartheta \vartheta' \rangle}{\partial \varrho_k} \zeta_k + \frac{f_\theta \theta^2}{r^2} \frac{\partial}{\partial r} (r^3 \lambda) \quad (23)$$

where $r^2 = \varrho_1^2 + \varrho_2^2 + \varrho_3^2$. For what concerns G_3 , it gives null contribution, as in isotropic turbulence, $\langle \vartheta \vartheta' \omega_i \rangle$ is a function of $r = |\mathbf{r}|$ and this implies that $G_3 \equiv 0$. Therefore G is

$$G = -\theta^2 \frac{\partial f_\theta}{\partial r} \lambda r - \frac{\partial \langle \vartheta \vartheta' \rangle}{\partial \varrho_k} \zeta_k \quad (24)$$

The first term of Eq. (24) is the consequence of $\lambda > 0$ and provides partially the mechanism of temperature cascade, giving a flow from small to big scales. The second one, related to the other two Lyapunov exponents, goes against the previous mechanism of cascade, being $\Delta \mathbf{u}$ solenoidal, thus it will have the opposite sign of the first term. To obtain the second term, observe that into Eqs. (17), $\lambda_2 = \lambda_3 = \lambda_\zeta = -\lambda/2$, and this leads to

$$\frac{\partial \langle \vartheta \vartheta' \rangle}{\partial \varrho_2} \zeta_2 = \frac{3}{2} \frac{\partial f_\theta}{\partial r} \lambda \theta^2 \frac{\varrho_2^2}{r}, \quad \frac{\partial \langle \vartheta \vartheta' \rangle}{\partial \varrho_3} \zeta_3 = \frac{3}{2} \frac{\partial f_\theta}{\partial r} \lambda \theta^2 \frac{\varrho_3^2}{r} \quad (25)$$

Moreover, because of the isotropy $\partial \langle \vartheta \vartheta' \rangle / \partial \varrho_1 \hat{\zeta}_1$ must be of the form

$$\frac{\partial \langle \vartheta \vartheta' \rangle}{\partial \varrho_1} \hat{\zeta}_1 = \frac{3}{2} \frac{\partial f_\theta}{\partial r} \lambda \theta^2 \frac{\varrho_1^2}{r} \quad (26)$$

Hence

$$\frac{\partial \langle \vartheta \vartheta' \rangle}{\partial r_k} \zeta_k = -\frac{3}{2} \theta^2 \frac{\partial f_\theta}{\partial r} \lambda r \quad (27)$$

Accordingly, the analytical expression of G is

$$G(r) = \frac{\theta^2}{2} \frac{\partial f_\theta}{\partial r} \lambda r = \theta^2 u \sqrt{\frac{1-f}{2}} \frac{\partial f_\theta}{\partial r} \quad (28)$$

Equation (28), representing the thermal energy cascade, gives the proposed closure of the Corrsin equation and expresses the combined effect of temperature and velocity correlations. Its main asset with respect to the other models is that it is not based on the phenomenological assumption, but is derived from a specific finite-scale Lyapunov analysis, under the assumption of homogeneous isotropic turbulence. Equation (28) preserves θ and provides the mechanism of the thermal energy transfer. This latter consists of a flow of the thermal energy from large to small scales which only redistributes the thermal energy between wavelengths and whose effectiveness depends upon f and f_θ . According to the Lyapunov analysis of de Divitiis (2010), this mechanism can be viewed in the following manner. If, at an initial time t_0 , a toroidal material volume $\Sigma(t_0)$ is considered, which includes an assigned amount of thermal energy, its geometry and position change according to the fluid motion, and its dimensions will vary to preserve the volume. Choosing Σ in such a way that the maximal dimension of the toroid R increases

with t , the finite-scale Lyapunov analysis leads to $R \approx R(t_0)e^{\lambda(t-t_0)}$. The thermal energy, initially enclosed into $\Sigma(t_0)$, at the end of the fluctuation is contained into $\Sigma(t)$ whose dimensions are changed with respect to the initial time t_0 . Therefore, the thermal energy is transferred from large to small scales, resulting enclosed in a more thin toroid.

4. Formulation of the problem

At this stage of the analysis, the problem for determining f and f_θ is formulated through the von Kármán-Howarth and Corrsin equations, which are here reported

$$\frac{\partial f}{\partial t} = \frac{K(r)}{u^2} + 2\nu \left(\frac{\partial^2 f}{\partial r^2} + \frac{4}{r} \frac{\partial f}{\partial r} \right) + \frac{10\nu}{\lambda_T^2} f \quad (29)$$

$$\frac{\partial f_\theta}{\partial t} = \frac{G(r)}{\theta^2} + 2\chi \left(\frac{\partial^2 f_\theta}{\partial r^2} + \frac{2}{r} \frac{\partial f_\theta}{\partial r} \right) + \frac{12\chi}{\lambda_\theta^2} f_\theta \quad (30)$$

whose boundary conditions are

$$\begin{aligned} f(0) &= 1, & \lim_{r \rightarrow \infty} f(r) &= 0, \\ f_\theta(0) &= 1, & \lim_{r \rightarrow \infty} f_\theta(r) &= 0 \end{aligned} \quad (31)$$

K and G are expressed by means of the finite scale Lyapunov analysis (Eqs. (93) and (28))

$$\begin{aligned} K &= u^3 \sqrt{\frac{1-f}{2}} \frac{\partial f}{\partial r}, \\ G &= \theta^2 u \sqrt{\frac{1-f}{2}} \frac{\partial f_\theta}{\partial r} \end{aligned} \quad (32)$$

As k and ν are considered to be assigned quantities, according to Eqs. (29) and (30), f_θ is related to f , whereas f does not depend upon f_θ .

The energy spectrum $E(\kappa)$ and the transfer function $T(\kappa)$ are the Fourier Transforms of f and K , respectively (Batchelor (1953))

$$\begin{bmatrix} E(\kappa) \\ T(\kappa) \end{bmatrix} = \frac{1}{\pi} \int_0^\infty \begin{bmatrix} u^2 f(r) \\ K(r) \end{bmatrix} \kappa^2 r^2 \left(\frac{\sin \kappa r}{\kappa r} - \cos \kappa r \right) dr \quad (33)$$

accordingly, the temperature spectrum $\Theta(\kappa)$ and the temperature transfer function $\Gamma(\kappa)$ are here calculated as (Ogura (1958))

$$\begin{bmatrix} \Theta(\kappa) \\ \Gamma(\kappa) \end{bmatrix} = \frac{2}{\pi} \int_0^\infty \begin{bmatrix} \theta^2 f_\theta(r) \\ G(r) \end{bmatrix} \kappa r \sin \kappa r \, dr \quad (34)$$

in such a way that

$$f_\theta(r) = \int_0^\infty \Theta(\kappa) \frac{\sin \kappa r}{\kappa r} \, d\kappa, \quad G(r) = \int_0^\infty \Gamma(\kappa) \frac{\sin \kappa r}{\kappa r} \, d\kappa \quad (35)$$

and

$$\int_0^\infty \Theta(\kappa) \, d\kappa = \theta^2, \quad \int_0^\infty \Gamma(\kappa) \, d\kappa = 0 \quad (36)$$

5. Lyapunov analysis of the temperature fluctuations

The proposed procedure for calculating the temperature fluctuations is based on the Lyapunov analysis of the fluid strain just proposed by de Divitiis (2010), and on the adoption of Eq. (1).

In order to obtain the temperature fluctuation, consider now the relative motion between two contiguous particles, expressed by the infinitesimal separation vector $d\mathbf{x}$ which obeys to the equation

$$d\dot{\mathbf{x}} = \nabla \mathbf{u} \, d\mathbf{x} \quad (37)$$

where $d\mathbf{x}$ varies according to the velocity gradient which in turn follows the Navier-Stokes equations. As observed by de Divitiis (2010), in turbulence, $d\mathbf{x}$ is much faster than the fluid state variables, and the Lyapunov analysis of

Eq. (37) provides the expression of the local deformation in terms of maximal Lyapunov exponent $\Lambda \equiv \lambda(0) > 0$

$$\frac{\partial \mathbf{x}}{\partial \mathbf{x}_0} \approx e^{\Lambda(t-t_0)} \quad (38)$$

Now, the map $\chi : \mathbf{x}_0 \rightarrow \mathbf{x}$, is the function which determines the current position \mathbf{x} of a fluid particle located at the referential position \mathbf{x}_0 at $t = t_0$ (Truesdell (1977)). Equation (1) can be written in terms of the referential position \mathbf{x}_0

$$\frac{\partial \vartheta}{\partial t} = \left(-\frac{\partial \vartheta}{\partial x_{0p}} u_h + \chi \frac{\partial^2 \vartheta}{\partial x_{0p} \partial x_{0q}} \frac{\partial x_{0q}}{\partial x_h} \right) \frac{\partial x_{0p}}{\partial x_h} \quad (39)$$

The adoption of the referential coordinates allows to factorize the temperature fluctuation and to express it in Lyapunov exponential form of the local deformation. As this deformation is assumed to be much more rapid than $\partial \vartheta / \partial x_{0p} u_h$ and $\chi \partial^2 \vartheta / \partial x_{0p} \partial x_{0q}$, the temperature fluctuation can be obtained integrating Eq. (39) with respect to the time, where $\partial \vartheta / \partial x_{0p} u_h$ and $\chi \partial^2 \vartheta / \partial x_{0p} \partial x_{0q}$ are considered to be constant

$$\vartheta \approx \frac{1}{\Lambda} \left(-\frac{\partial \vartheta}{\partial x_{0p}} u_h + \chi \frac{\partial^2 \vartheta}{\partial x_{0p} \partial x_{0q}} \right)_{t=t_0} \approx \frac{1}{\Lambda} \left(\frac{\partial \vartheta}{\partial t} \right)_{t=t_0} \quad (40)$$

This assumption is justified by the fact that, according to the classical formulation of motion of continuum media (Truesdell (1977)), $\partial \vartheta / \partial x_{0p} u_h$ and $\chi \partial^2 \vartheta / \partial x_{0p} \partial x_{0q}$ are smooth functions of t -at least during the period of a fluctuation- whereas the fluid deformation varies very rapidly according to Eqs. (37)-(38).

6. Statistical analysis of temperature derivative

As explained in this section, the Lyapunov analysis of the local deformation and some plausible assumptions about the statistics of \mathbf{u} and ϑ , lead to determine the distribution function of $\partial \vartheta / \partial r$ and all its dimensionless statistical moments.

The statistical properties of $\partial \vartheta / \partial r$, are here investigated expressing velocity and temperature through the following canonical decomposition (Ventsel (1973))

$$\mathbf{u} = \sum_k \hat{\mathbf{U}}_k \xi_k, \quad \vartheta = \sum_k \hat{\Theta}_k \xi_k \quad (41)$$

where $\hat{\mathbf{U}}_k$ and $\hat{\Theta}_k$ are proper coordinate functions of t and \mathbf{x} , and ξ_k ($k = 1, 2, \dots$) are dimensionless independent stochastic variables which satisfy

$$\langle \xi_k \rangle = 0, \quad \langle \xi_i \xi_j \rangle = \delta_{ij}, \quad \langle \xi_i \xi_j \xi_k \rangle = \varpi_{ijk} p, \quad |p| \gg 1, \quad \langle \xi_k^4 \rangle = O(1) \quad (42)$$

where $\varpi_{ijk} = 1$ for $i = j = k$, else $\varpi_{ijk} = 0$. It is worth to remark that the variables ξ_k are properly chosen in such a way that they express the mechanism of cascade for both velocity and temperature, through the condition $|p| \gg 1$.

The dimensionless temperature fluctuation $\hat{\vartheta}$ is obtained in terms of ξ_k substituting Eq. (41) into Eq. (40)

$$\hat{\vartheta} = \sum_{ij} A_{ij} \xi_i \xi_j + \frac{1}{Pe} \sum_k b_k \xi_k \quad (43)$$

where $r = \hat{r} \lambda_T$, $\vartheta = \hat{\vartheta} \theta$, whereas $Pe = R Pr$ and $R = u \lambda_T / \nu$ are Péclet and Reynolds numbers referred to the Taylor scale, and $Pr = \nu / \chi$ is the fluid Prandtl number, therefore $\sum_{ij} A_{ij} \xi_i \xi_j$ and $1/Pe \sum_k b_k \xi_k$ arise from convective term and fluid conduction, respectively. Now, thanks to the local isotropy, both \mathbf{u} and ϑ are two gaussian stochastic variables (Ventsel (1973), Lehmann (1999)), accordingly, ξ_k satisfy the Lindeberg condition, a very general necessary and sufficient condition for satisfying the central limit theorem (Lehmann (1999)). This condition does not apply to $\partial \vartheta / \partial r \equiv \lim_{r \rightarrow 0} \Delta \vartheta / r$. In fact, as $\Delta \vartheta$ is the difference between two correlated gaussian variables, its PDF could be a non gaussian distribution function. To obtain this PDF, the fluctuation $\partial \hat{\vartheta} / \partial \hat{r}$ is first calculated in function of ξ_k

$$\frac{\partial \hat{\vartheta}}{\partial \hat{r}} = \sum_{ij} \frac{\partial A_{ij}}{\partial \hat{r}} \xi_i \xi_j + \frac{1}{Pe} \sum_k \frac{\partial b_k}{\partial \hat{r}} \xi_k \equiv L + S + P + N \quad (44)$$

This fluctuation consists of the contributions L , S , P and N , appearing into Eq. (44): in particular, L is the sum of all linear terms due to the fluid conductivity, $S \equiv S_{ij} \xi_i \xi_j$ is the sum of all semidefinite bilinear forms arising from the convective term, whereas P and N are, respectively, the sums of definite positive and negative quadratic forms, which derive from the convective term. The quantity $L + S$ tends to a gaussian random variable being the sum of statistically orthogonal terms, while P and N do not, as

they are linear combinations of squares (Madow (1940), Lehmann (1999)). Their general expressions are

$$\begin{aligned} P &= P_0 + \eta_1 + \eta_2^2 \\ N &= N_0 + \zeta_1 - \zeta_2^2 \end{aligned} \tag{45}$$

where P_0 and N_0 are constants, and η_1, η_2, ζ_1 and ζ_2 are four different centered random gaussian variables which are mutually uncorrelated thanks to the hypotheses of fully developed flow and isotropy. Therefore, the longitudinal fluctuation of the temperature derivative can be written as

$$\frac{\partial \hat{\vartheta}}{\partial \hat{r}} = \psi_1 \xi + (\psi_2 (\eta^2 - 1) - \psi_3 (\zeta^2 - 1)) \tag{46}$$

where ξ, η and ζ are independent centered random variables which exhibit gaussian PDF with $\langle \xi^2 \rangle = \langle \eta^2 \rangle = \langle \zeta^2 \rangle = 1$, and ψ_1, ψ_2 and ψ_3 are given quantities. Due to the isotropy, the skewness of $\partial \hat{\vartheta} / \partial \hat{r}$ must be equal to zero, thus $\psi_2 = \psi_3$, and

$$\frac{\partial \hat{\vartheta}}{\partial \hat{r}} = \psi_1 \xi + \psi_2 (\eta^2 - \zeta^2) \tag{47}$$

Furthermore, comparing the terms of Eqs. (47) and (44), we obtain that ψ_1 and ψ_2 are related each other and that their ratio $\psi = \psi_1 / \psi_2$ depends on the Péclet number

$$\frac{4\psi_2^2}{\psi_1^2} = \frac{\langle (P + N)^2 \rangle}{\langle (S_{ij} \xi_i \xi_j + 1/Pe \partial b_k / \partial \hat{r} \xi_k)^2 \rangle} \tag{48}$$

Taking into account the properties (42) of ξ_k ($\langle \xi_k^3 \rangle \gg 1$, $\langle \xi_k^4 \rangle = O(1)$), and in view of Eq. (48), we found

$$\psi \equiv \frac{\psi_2}{\psi_1} = C \sqrt{Pe} \tag{49}$$

where C is a proper constant which has to be identified. Hence, the dimensionless longitudinal temperature derivative is

$$\frac{\vartheta_r}{\sqrt{\langle \vartheta_r^2 \rangle}} = \frac{\xi + \psi (\eta^2 - \zeta^2)}{\sqrt{1 + 4\psi^2}} \tag{50}$$

In order to identify C , observe that Eq. (50) is formally similar to the expression of the longitudinal velocity derivative $\partial u_r / \partial r$ obtained in de Divitiis (2010) (see also the appendix)

$$\frac{\partial u_r / \partial r}{\sqrt{\langle (\partial u_r / \partial r)^2 \rangle}} = \frac{\xi_u + \psi_u (\chi (\eta_u^2 - 1) - (\zeta_u^2 - 1))}{\sqrt{1 + 2\psi_u^2 (1 + \chi^2)}} \quad (51)$$

ξ_u , η_u and ζ_u are independent centered gaussian random variables with $\langle \xi_u^2 \rangle = \langle \eta_u^2 \rangle = \langle \zeta_u^2 \rangle = 1$, and

$$\psi_u(R) = \sqrt{\frac{R}{15\sqrt{15}}} \hat{\psi}_u(0), \quad \hat{\psi}_u(0) = O(1), \quad \chi = \chi(R) = O(1) \quad (52)$$

$\chi \neq 1$ provides a negative skewness of $\partial u_r / \partial r$, whereas $\hat{\psi}_u(0) \simeq 1.075$ is determined through an approximate estimation of the critical value of R (de Divitiis (2010)). Now, when $Pr = 1$, it is reasonable to assume that the ratio between linear and quadratic terms of Eq. (50) is equal to that of the corresponding terms of Eq. (51). Accordingly, $\psi \simeq \psi_u$ and this identifies an approximate value of C

$$C \approx \frac{\hat{\psi}_u(0)}{15^{3/4}} \simeq 0.141 \quad (53)$$

The distribution function of the temperature derivatives is thus expressed through the Frobenius-Perron equation

$$F(\vartheta'_r) = \int_{\xi} \int_{\eta} \int_{\zeta} p(\xi) p(\eta) p(\zeta) \delta(\vartheta'_r - \vartheta_r(\xi, \eta, \zeta)) d\xi d\eta d\zeta \quad (54)$$

where $\vartheta_r(\xi, \eta, \zeta)$ is determined with Eq. (50), δ is the Dirac delta and p is a centered gaussian PDF with standard deviation equal to one.

Finally, the dimensionless statistical moments of ϑ_r are easily calculated considering that ξ , η and ζ are independent gaussian variables

$$H_n \equiv \frac{\langle \vartheta_r^n \rangle}{\langle \vartheta_r^2 \rangle^{n/2}} = \frac{1}{(1 + 4\psi^2)^{n/2}} \sum_{k=0}^n \binom{n}{k} \psi^k \langle \xi^{n-k} \rangle \langle (\eta^2 - \zeta^2)^k \rangle \quad (55)$$

It is worth to remark that, for non-isotropic turbulence or in more complex cases with boundary conditions, the stochastic variables ξ_k could not satisfy

the Lindeberg condition, thus ϑ will be not distributed following a Gaussian PDF, and Eq. (50) changes its analytical form and can incorporate more intermittent terms (Lehmann (1999)) which give the deviation with respect to the isotropic turbulence. Hence, the absolute statistical moments of ϑ_r will be greater than those calculated with Eq.(55), indicating that, in a more complex situation than the isotropic turbulence, the intermittency of ϑ_r can be significantly stronger.

7. Self-Similar temperature spectrum

An ordinary differential equation which describes the spatial evolution of f_θ is now derived from Eq. (30), adopting the hypothesis of self-similarity of von Kármán & Lin (1949), George (1988)-George (1992), and using the proposed closure of the Corrsin equation.

Far from the initial condition, the simultaneous effect of temperature and velocity cascade with the fluid conductivity and viscosity, acts keeping f_θ similar in the time. This is the idea of self-preserving correlation function which was originally introduced by von Kármán (see von Kármán & Lin (1949) and reference therein) for what concerns the velocity correlation, and thereafter adopted by George (1988)-George (1992) for studying the temperature spectrum. According to George (1988)-George (1992), the self-similar temperature correlation can be scaled with respect to $\lambda_T(t)$, thus

$$f_\theta = f_\theta(\hat{r}), \text{ where } \hat{r} = r/\lambda_T(t) \quad (56)$$

Substituting Eq. (56) into Eq. (30), we obtain

$$-\frac{df_\theta}{d\hat{r}} \frac{\hat{r}}{u} \frac{d\lambda_T}{dt} = \sqrt{\frac{1-f}{2}} \frac{df_\theta}{d\hat{r}} + \frac{2}{R Pr} \left(\frac{d^2 f_\theta}{d\hat{r}^2} + \frac{2}{\hat{r}} \frac{df_\theta}{d\hat{r}} \right) + \frac{12}{R Pr} \left(\frac{\lambda_T}{\lambda_\theta} \right)^2 f_\theta \quad (57)$$

Therefore, the boundary problem given by Eqs. (30) and (31). is here reduced to an ordinary differential equation of the second order in the variable r , Equation (57) is a non-linear equation whose coefficients vary according to Eq. (100) and

$$\frac{du^2}{dt} = -\frac{10\nu u^2}{\lambda_T^2}, \quad \frac{d\theta^2}{dt} = -\frac{12k\theta^2}{\lambda_\theta^2} \quad (58)$$

Now, if the self-similarity is assumed, all the coefficients of Eq. (57) do not vary with the time (von Kármán & Howarth (1938)-von Kármán & Lin (1949), George (1988)-George (1992)), thus

$$R = \text{const}, \quad \frac{1}{u} \frac{d\lambda_T}{dt} = \text{const}, \quad \frac{\lambda_\theta}{\lambda_T} = \text{const} \quad (59)$$

As λ_T follows Eq. (100), λ_θ is obtained from the constancy of λ_θ/λ_T

$$\lambda_\theta(t) = \lambda_\theta(0) \sqrt{1 + 10\nu t/\lambda_T^2(0)}. \quad (60)$$

Thus, according to Warhaft & Lumley (1978) and George (1988)-George (1992), the microscales λ_T , λ_θ and the rates $d\theta^2/dt$ and du^2/dt , depend on the initial conditions of temperature and kinetic energy spectra. Taking into account Eq. (100), we obtain

$$\frac{1}{u} \frac{d\lambda_T}{dt} = \frac{5}{R} \quad (61)$$

Therefore, $f_\theta(\hat{r})$ obeys to the following non-linear ordinary differential equation

$$\frac{5}{R} \frac{df_\theta}{d\hat{r}} \hat{r} + \sqrt{\frac{1-f}{2}} \frac{df_\theta}{d\hat{r}} + \frac{2}{R Pr} \left(\frac{d^2 f_\theta}{d\hat{r}^2} + \frac{2}{\hat{r}} \frac{df_\theta}{d\hat{r}} \right) + \frac{12}{R Pr} \left(\frac{\lambda_T}{\lambda_\theta} \right)^2 f_\theta = 0 \quad (62)$$

The self-similar solutions are searched over the whole range of \hat{r} , but for the dimensionless distances whose order magnitude exceed R . This corresponds to assume the self-similarity for all the frequencies of the energy spectrum, with the exception of the lowest ones (von Kármán & Howarth (1938), von Kármán & Lin (1949)). Accordingly, $\partial f_\theta/\partial t$, can be neglected with respect to the other terms

$$R Pr \sqrt{\frac{1-f}{2}} \frac{df_\theta}{d\hat{r}} + 2 \left(\frac{d^2 f_\theta}{d\hat{r}^2} + \frac{2}{\hat{r}} \frac{df_\theta}{d\hat{r}} \right) + 12 \left(\frac{\lambda_T}{\lambda_\theta} \right)^2 f_\theta = 0 \quad (63)$$

The boundary conditions of Eq. (63) arise from Eqs. (11)

$$f_\theta(0) = 1 \quad (64)$$

$$\lim_{\hat{r} \rightarrow \infty} f_\theta(\hat{r}) = 0 \quad (65)$$

For $\hat{r} = 0$, Eq. (63) gives

$$\frac{d^2 f_\theta(0)}{d\hat{r}^2} = -2 \left(\frac{\lambda_T}{\lambda_\theta} \right)^2 \quad (66)$$

To determine f_θ , λ_T/λ_θ must be first specified into Eq. (63). This is determined using again the self-similarity of f and f_θ , thus λ_T/λ_θ is calculated by substituting Eq. (60) into Eq. (58) and integrating this latter from $t = 0$ to t

$$\ln \left(\frac{\theta(t)}{\theta(0)} \right) = -\frac{3}{5} \left(\frac{\lambda_T}{\lambda_\theta} \right)^2 \frac{1}{Pr} \ln \left(1 + \frac{10\nu}{\lambda_T(0)^2} t \right) \quad (67)$$

whereas the velocity standard deviation is

$$\ln \left(\frac{u(t)}{u(0)} \right) = -\frac{1}{2} \ln \left(1 + \frac{10\nu}{\lambda_T(0)^2} t \right) \quad (68)$$

The full self-similarity (mechanical and thermal) occurs when θ and u are proportional each other

$$\frac{\theta(t)}{\theta(0)} = \frac{u(t)}{u(0)} \quad (69)$$

The value of λ_T/λ_θ satisfying this condition depends on the Prandtl's number, and is calculated with Eq. (67)

$$\frac{\lambda_\theta}{\lambda_T} = \sqrt{\frac{6}{5} \frac{1}{Pr}} \quad (70)$$

Accordingly, $f_\theta''(0)$ is related to Pr

$$\frac{d^2 f_\theta(0)}{d\hat{r}^2} = -\frac{5}{3} Pr \quad (71)$$

This result, in agreement with Corrsin (JAP 1951), George (1988), George (1992), expresses a further link between f and f_θ only in the case of self-similarity.

Observe that the solutions $f_\theta \in C^2[0, \infty)$ of Eq. (63) with $df_\theta/d\hat{r}(0) = 0$ and $Pr \neq 0$, tend to zero as $r \rightarrow \infty$, thus the boundary condition (65) can be replaced by the following condition in the origin

$$\frac{df_\theta(0)}{d\hat{r}} = 0 \quad (72)$$

Therefore, the boundary problem represented by Eqs. (63), (64) and (65), is reduced to the following initial condition problem written in the Cauchy's normal form

$$\begin{aligned}\frac{df_\theta}{d\hat{r}} &= F_\theta \\ \frac{dF_\theta}{d\hat{r}} &= -5 \, Pr \, f_\theta - \left(\frac{R \, Pr}{2} \sqrt{\frac{1-f}{2}} + \frac{2}{\hat{r}} \right) F_\theta\end{aligned}\tag{73}$$

the initial condition of which is

$$f_\theta(0) = 1, \quad F_\theta(0) = 0\tag{74}$$

In conclusion, the self-similar functions f and f_θ are calculated as the solutions of the ordinary differential system (102) and (73) with the initial conditions (103) and (74).

8. Results and Discussion

The self-similar temperature and longitudinal velocity correlations are here calculated with Eqs. (102) and (73), for several values of R and Pr .

The case with $Pr \rightarrow 0$ (infinitely conductive fluid) is first considered. This is a limit case of the differential system (73)-(74) corresponding to the following equation

$$\frac{d^2 f_\theta}{d\hat{r}^2} + \frac{2}{\hat{r}} \frac{df_\theta}{d\hat{r}} = 0\tag{75}$$

which does not admit analytical solutions for the boundary conditions (31).

Conversely, the case $Pr \rightarrow \infty$ ($\nu \gg \chi$) gives the following result

$$\lim_{\hat{r} \rightarrow 0} \frac{d^2 f_\theta}{d\hat{r}^2} = -\infty\tag{76}$$

which expresses the behavior of the temperature correlation near the origin, whereas, for $\hat{r} > 0$, f_θ is obtained solving Eqs. (73)-(74) by quadrature, in terms of f

$$\ln f_\theta(\hat{r}) = -\frac{10\sqrt{2}}{R} \int_0^{\hat{r}} \frac{d\xi}{\sqrt{1-f(\xi)}}\tag{77}$$

This equation states that, if R is large enough and

$$f \simeq 1 - \left(\frac{r}{L_u} \right)^{2/3}, \quad \text{then also} \quad f_\theta \simeq 1 - \left(\frac{r}{L_\theta} \right)^{2/3} \quad (78)$$

where $L_\theta = L_u$ are length scales proportional to λ_T

$$L_u = L_\theta = \frac{R}{15\sqrt{2}}\lambda_T, \quad (79)$$

Therefore, in case of self-similarity, with $Pr \rightarrow \infty$, the ratios between the scales are

$$\frac{L_\theta}{L_u} = 1, \quad \frac{\lambda_\theta}{\lambda_T} = 0 \quad (80)$$

When R and Pr change, the ratios between the scales vary depending on the combined values of R and Pr , therefore quite different situations occur.

In order to study the influence of R and Pr on f_θ , Eqs. (102) and (73) are numerically solved for different values of R and Pr . The Reynolds number is assumed to be $R=50, 100$ and 300 , whereas Pr ranges from 0.001 , to 10 .

Figure 3 shows f (dashed lines) and f_θ (solid line) in such these conditions. The temperature correlation, related to f by means of the mechanism of temperature cascade (see Eq. (28)), is furthermore linked to f by self-similarity (70). Therefore, for assigned values of R and λ_T , the Corrsin microscale decreases with Pr and the curves of f_θ seem to collapse into a single diagram when $Pr \rightarrow \infty$. On the contrary, small values of Pr , representing high thermal conductivity, determine large scales of variations of f_θ and p_* . In particular, the case $R = 50$ is first considered. For $Pr = 0.001$, f_θ exhibits oscillations whose amplitude decreases when \hat{r} rises. As Pr increases, the oscillations magnitude diminishes, and for $0.01 < Pr < 0.1$ these oscillations vanish, being $f_\theta > 0$, whereas the integral scales and the Corrsin length diminish. The case $R = 100$ differs from the previous one. In fact, the higher value of R determines sizable reduction of the oscillations, whereas the integral scales of f and f_θ are greater than the previous ones. Next, for $R = 300$, the integral scales increase again, resulting $f_\theta > 0$ a monotonic function of r for each value of Pr .

Accordingly, also the triple correlation temperature-velocity p_* varies with R and Pr . For $R = 50$, small values of Pr (0.001) cause large scales of variations and sizable oscillations of p_* , whereas higher Prandtl numbers

produce the loss of these oscillations and a reduction of the length scales and of $|p_*|$. Increasing R ($R = 100$ and 300), the length scales rise, the oscillations disappear, and a reduction of $|p_*|_{MAX}$ is observed. The Prandtl's number acts on f_θ , in such a way that its increment causes a diminishing of the length scales and of $|p_*|$. When $Pr = 0.7$ and 1 , the obtained results agree very well with the classical experiments of Mills et al (1958) which regards the turbulence behind heated grid, in the sense that, f_θ is roughly equal to f , whereas p_* and the triple velocity correlation k exhibit the same order of magnitude.

As far as the velocity correlations, $f(r)$ and $k(r)$, are concerned, these agree with the results of de Divitiis (2011).

The properties of $\Theta(\kappa)$ are the consequence of the temperature correlations. These spectra, calculated with Eq. (33) and (34), are depicted in Fig. 4. The variations of $\Theta(\kappa)$ with R and Pr are quite peculiar. In any case, according to Eq. (28), $n \rightarrow 2$, as $\kappa \rightarrow 0$. For $Pr = 0.001$, when R ranges from 50 to 300, the temperature spectrum shows essentially two regions, which correspond to two different scaling laws $\Theta(\kappa) \approx \kappa^n$ (see also Fig. 5): one near the origin where $n \simeq 2$, and the other one, at higher κ , where $-17/3 < n < -11/3$, (value very close to $-13/3$). Between these regions, the exponent n varies rapidly at low Reynolds number, whereas when R increases, n exhibits more gradual variations. The value of $n \approx -13/3$ here calculated, is in between the exponent proposed by Batchelor et al (1959) ($-17/3$) and the value determined by Rogallo et al (1989) ($-11/3$) with the numerical simulations. Increasing κ , n strongly diminishes, and $\Theta(\kappa)$ does not show scaling law. When $Pr=0.01$, the three curves intersect each other for $n \simeq -5/3$, where these have inflection points. The width of the interval where $-17/3 < n < -11/3$ diminishes, in particular now $-17/3 < n < -13/3$, value in agreement with Batchelor et al (1959). For $Pr=0.1$, the previous scaling law with $n \approx -13/3$ vanishes, whereas for $R = 50$ and 100 , n changes with κ and $\Theta(\kappa)$ does not show a noticeable scaling law. When $R = 300$, the born of a small region in which n has an inflection point is observed for $n \approx -5/3$. For $Pr = 0.7$ and 1 , with $R = 300$, the width of this interval is increased, whereas at $Pr = 10$, and $R = 300$, we observe two regions: one interval where n has a local minimum with $n \simeq -5/3$, and the other one where n exhibits a relative maximum, with $n \simeq -1$. For larger κ , n diminishes and the scaling laws disappear.

Figure 4 reports also (on the bottom) the spectra $\Gamma(\kappa)$ (solid lines) and $T(\kappa)$ (dashed lines) which describe the mechanism of kinetic energy and

temperature cascade. As these latter do not modify the value of θ and u , $\int_0^\infty T(\kappa)d\kappa \equiv 0$, and $\int_0^\infty \Gamma(\kappa)d\kappa \equiv 0$.

The presence of the scaling law $n \simeq -5/3$ agrees with the theoretical arguments of Corrsin (JAP 1951), Obukhov (1949) (see also Mydlarski & Warhaft (1998), Donzis et al (2010) and references therein), where in the limit of high R and Pr

$$\Theta(\kappa) = C_\theta \epsilon^{-1/3} \epsilon_\theta \kappa^{-5/3}, \quad (81)$$

in the inertial-convective range, being

$$\epsilon_\theta = 12\chi \frac{\theta^2}{\lambda_\theta^2}, \quad \epsilon = 15\nu \frac{u^2}{\lambda_T^2} \quad (82)$$

and C_θ is the so-called Corrsin-Obukhov constant, a quantity of the order of the unity. This study identifies C_θ by means of the obtained results, analysing the quantity

$$F_C(\kappa) = \Theta(\kappa) \epsilon^{1/3} \kappa^{5/3} \epsilon_\theta^{-1} \quad (83)$$

-here called Corrsin function- in terms of κ , Re and Pr . This is calculated as $C_\theta = F_C(\kappa)$ in the range where $F_C(\kappa)$ is about constant. A different Corrsin-Obukhov constant $C_{\theta 1}$ can be also defined with respect to the one dimensional spectrum

$$\frac{d\Theta_1}{d\kappa_1}(\kappa_1) = -\frac{\Theta(\kappa_1)}{2\kappa_1}, \text{ where } \Theta_1(\kappa) = C_{\theta 1} \epsilon^{-1/3} \epsilon_\theta \kappa^{-5/3} \text{ and } C_{\theta 1} = 0.3C_\theta \quad (84)$$

Figure 6 reports F_C in terms of κ for different values of R and Pr . For $Pr = 0.01$, at relatively small Reynolds number, the temperature spectrum does not follow $\kappa^{-5/3}$, thus C_θ is not defined, whereas at $R = 300$ the diagram shows a region with a local maximum where the variations of F_C are relatively small. This maximum can identify the value of C_θ which results to be about 1.5 ($C_{\theta 1} \simeq 0.45$). For $Pr = 0.1$, the larger scaling interval, implies a wider range in which $F_C(\kappa) \simeq \text{const}$, for each Reynolds number, resulting now $C_\theta \simeq 1.8$ ($C_{\theta 1} \simeq 0.54$). When $Pr = O(1)$ (0.7 in the figure), the scaling law κ^{-1} , determines that F_C slightly rises with κ , ranging from 1.4 to 2 ($C_{\theta 1} \simeq 0.42 \div 0.58$), values comparable with Mydlarski & Warhaft (1998). For $Pr = 10$, the region with $\Theta \approx \kappa^{-1}$ increases, therefore F_C reveals sizable variations and C_θ can be defined only in a small range of κ .

The scaling law k^{-1} is in line with the theoretical arguments proposed in Batchelor (1959), where

$$\Theta(\kappa) = C_B \sqrt{\frac{\nu}{\epsilon}} \epsilon_\theta \kappa^{-1}, \quad (85)$$

in the viscous-convective range, being $C_B = O(1)$ the Batchelor's constant. The present analysis identifies C_B through the temperature spectra previously calculated. To this end, the following quantity

$$F_B(\kappa) = \Theta(\kappa) \kappa \sqrt{\frac{\epsilon}{\nu}} \epsilon_\theta^{-1} \quad (86)$$

here called Batchelor's function, is considered for several κ , Re and Pr . This constant is here calculated as $C_\theta = F_B(\kappa)$ in the region of κ where $F_B(\kappa) \approx \text{const}$ (or at least exhibits a plateau). Also the Batchelor's constant C_{B1} can be defined with respect to the one dimensional spectrum (84)

$$\Theta_1(\kappa_1) = C_{B1} \sqrt{\frac{\nu}{\epsilon}} \epsilon_\theta \kappa_1^{-1}, \quad \text{where } C_{B1} = 0.5 C_B \quad (87)$$

In Fig. 7, F_B is represented versus κ for different values of Re and Pr . It is apparent that $F_B \approx \text{const}$ when Pr is high enough. For $Pr = 10$, when $R = 50, 100$ and 300 , the corresponding values of C_B are about 5, 7 and 8 (that is $C_{B1} \simeq 2.5, 3.5$ and 4) and this occurs for $1 < \kappa < 10$. These values are in quite good agreement with Donzis et al (2010) (and references therein), and are consistent with the experiments of Grant et al (1968) and Oakey (1982) which deal with the temperature spectrum observed in ocean.

Next, in order to analyse the statistics of the temperature derivative, the PDF of $\partial\vartheta/\partial r$ is calculated with Eqs. (54) and (50), for different values of the parameter $\psi = C\sqrt{Pr R}$. This PDF is obtained with sequences of the variables ξ , η and ζ , each generated by a gaussian random numbers generator. The distribution function is then calculated through the statistical elaboration of these data and Eq. (50). The corresponding results are in Fig. 8, where the PDF is shown in terms of the dimensionless abscissa

$$s = \frac{\partial\vartheta/\partial r}{\sqrt{\langle \partial\vartheta/\partial r^2 \rangle}} \quad (88)$$

These distribution functions are normalized, in order that their standard deviations are equal to the unity. These PDF are even functions of s and their tails rise with ψ in such a way that the intermittency of $\partial\vartheta/\partial r$ increases with ψ , according to Eq. (55). The PDFs shown in the figure are calculated for $\psi = 0, 0.25, 0.5, 1., 10., \infty$, in particular for $\psi = 10$ and ∞ , the curves are about overlapped. In order to study the influence of ψ on this intermittency and on the statistics of s , the flatness H_4 and the hyperflatness H_6 , defined as

$$H_4 = \frac{\langle s^4 \rangle}{\langle s^2 \rangle^2}, \quad H_6 = \frac{\langle s^6 \rangle}{\langle s^2 \rangle^3} \quad (89)$$

are also calculated. These are depicted in Fig. 8 in function of ψ . For $\psi = 0$, the PDF is gaussian which corresponds to $H_4 = 3$ and $H_6 = 15$. Increasing ψ , the non-linear fluctuations due to η and ζ , determine an increment of H_4 and H_6 that tend to the limits $H_4 = 9$ and $H_6 = 225$ when $\psi \rightarrow \infty$.

These results are compared with the experiments of Sreenivasan et al (1980) which in turn give the flatness of $\partial\vartheta/\partial r$. The value of C identified with this comparison of H_4 is $C \simeq 0.135$ against the value $C \approx 0.141$ here calculated in the proper section.

9. Conclusions

The finite scale Lyapunov theory is adopted to study the temperature fluctuations in homogeneous isotropic turbulence. This analysis leads to the closure of the Corrsin equation and provides the statistics of the temperature fluctuations.

The results, which represent a further application of the analysis presented in de Divitiis (2010) and de Divitiis (2011), are here obtained in the case of self-similar velocity and temperature fluctuations, and can be so summarized:

1. The energy equation, written using the referential coordinates and the Lyapunov analysis of the local deformation, allows to factorize the temperature fluctuation and to express it in Lyapunov exponential form of the local deformation.
2. The finite scale Lyapunov analysis provides an explanation of the physical mechanism of temperature cascade and gives the closure of the

Corrsin equation. This is a non-diffusive closure expressing G in terms of f and $\partial f_\theta / \partial r$.

3. This closure provides a mechanism of cascade which generates temperature spectra with different scaling laws, depending on R and Pr . In particular, for proper values of R and Pr , these spectra satisfy the Corrsin–Obukhov and Batchelor scaling laws in opportune regions of wave-numbers.
4. The Corrsin–Obukhov and Batchelor constants here identified with the proposed theory, agree with the different source from the literature.
5. The PDF of ϑ_r and the corresponding dimensionless moments are determined through a canonical decomposition of velocity and temperature in terms of random variables which describe the mechanism of cascade. This is a non-Gaussian PDF whose intermittency increases with R and Pr , in agreement with the experiments of the literature.

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11. Appendix

For sake of convenience, this section reports the main results of the finite scale Lyapunov analysis obtained by de Divitiis (2010) and de Divitiis (2011), which deal with the homogeneous isotropic turbulence.

11.1. Closure of the von Kármán-Howarth equation

For fully developed isotropic homogeneous turbulence, the pair correlation function f of the longitudinal velocity u_r , satisfies the von Kármán-Howarth equation (von Kármán & Howarth (1938))

$$\frac{\partial f}{\partial t} = \frac{K(r)}{u^2} + 2\nu \left(\frac{\partial^2 f}{\partial r^2} + \frac{4}{r} \frac{\partial f}{\partial r} \right) + \frac{10\nu}{\lambda_T^2} f \quad (90)$$

the boundary conditions of which are

$$\begin{aligned} f(0) &= 1, \\ \lim_{r \rightarrow \infty} f(r) &= 0 \end{aligned} \tag{91}$$

where r is the separation distance, $\lambda_T \equiv \sqrt{-1/f''(0)}$ is the Taylor scale, and u is the standard deviation of u_r , which satisfies the equation of the turbulent kinetic energy (von Kármán & Howarth (1938))

$$\frac{du^2}{dt} = -\frac{10\nu}{\lambda_T^2} u^2 \tag{92}$$

This equation, obtained putting $r = 0$ into Eq. (90), gives the rate of kinetic energy in function of u and λ_T (von Kármán & Howarth (1938), Batchelor (1953)). The function $K(r)$, related to the triple velocity correlation function, represents the effect of the inertia forces and expresses the mechanism of energy cascade. Thus, the von Kármán-Howarth equation provides the relationship between the statistical moments $\langle(\Delta u_r)^2\rangle$ and $\langle(\Delta u_r)^3\rangle$ in function of r , where Δu_r is the longitudinal velocity difference.

The Lyapunov theory proposed in de Divitiis (2010) leads to the closure of the von Kármán-Howarth equation, and expresses $K(r)$ in terms of f and $\partial f/\partial r$

$$K(r) = u^3 \sqrt{\frac{1-f}{2}} \frac{\partial f}{\partial r} \tag{93}$$

where $K(0) = 0$ represents the condition that K does not modify the fluid kinetic energy (von Kármán & Howarth (1938), Batchelor (1953)).

11.2. Statistics of the longitudinal velocity difference

Here, the results of de Divitiis (2010), dealing with the statistics of the longitudinal component of velocity difference Δu_r are recalled.

There, Δu_r is represented in terms of centered random variables

$$\frac{\Delta u_r}{\sqrt{\langle(\Delta u_r)^2\rangle}} = \frac{\xi_u + \psi_u (\chi(\eta_u^2 - 1) - (\zeta_u^2 - 1))}{\sqrt{1 + 2\psi_u^2(1 + \chi^2)}} \tag{94}$$

where ψ_u is a function of r and of the Taylor scale Reynolds number

$$\psi_u(\mathbf{r}, R) = \sqrt{\frac{R}{15\sqrt{15}}} \hat{\psi}_u(r) \quad (95)$$

$\psi_{u0} = \psi_u(R, 0)$, with $\hat{\psi}_u(0) = 1.075$, and $\chi \neq 1$ provides a nonzero skewness of Δu_r (de Divitiis (2010)).

Equation (94), arising from statistical considerations about the Fourier-transformed Navier-Stokes equations, expresses the internal structure of the fully developed isotropic turbulence, where ξ_u , η_u and ζ_u are independent centered random variables which exhibit the gaussian distribution functions $p(\xi_u)$, $p(\eta_u)$ and $p(\zeta_u)$ whose standard deviation is equal to the unity.

11.3. Self-Similarity in homogeneous isotropic turbulence

Now, the results of de Divitiis (2011) are briefly summarized, for what concerns the self-similarity of homogeneous isotropic turbulence. These results are based on the idea of self-preserving correlation function which was originally proposed by von Kármán & Howarth (1938)-von Kármán & Lin (1949): far from the initial condition, the combined effects of energy cascade and viscosity act keeping the velocity correlation function f and the energy spectrum $E(\kappa)$, similar in the time for large values of wavelengths, in particular in the inertial sub-range. The condition of self-preserving f , applied to Eq. (90), leads to the following ordinary differential equation

$$\sqrt{\frac{1-f}{2}} \frac{df}{d\hat{r}} + \frac{2}{R} \left(\frac{d^2 f}{d\hat{r}^2} + \frac{4}{\hat{r}} \frac{df}{d\hat{r}} \right) + \frac{10}{R} f = 0 \quad (96)$$

whose boundary conditions are from Eqs. (91) (von Kármán & Howarth (1938))

$$f(0) = 1, \quad (97)$$

$$\lim_{\hat{r} \rightarrow \infty} f(\hat{r}) = 0 \quad (98)$$

Into Eq. (96), $f = f(\hat{r})$, where \hat{r} is the dimensionless variable $\hat{r} = r/\lambda_T(t)$ which depends upon r and t , therefore

$$\frac{d^2 f}{d\hat{r}^2}(0) = -1 \quad (99)$$

Observe that Eq. (96) gives the self-similar f and E in the inertial range and for $\kappa \rightarrow \infty$ (small r). Although Eq. (96) does not describes the energy spectrum near the origin, $E(\kappa)$ satisfies the continuity equation, being $E \approx \kappa^4$ for $\kappa\lambda_T \ll 1$.

This similarity and the equation of the turbulent kinetic energy lead to the expressions of u and λ_T

$$\lambda_T(t) = \lambda_T(0) \sqrt{1 + 10\nu t / \lambda_T^2(0)}, \quad u(t) = \frac{u(0)}{\sqrt{1 + 10\nu t / \lambda_T^2(0)}}. \quad (100)$$

As the solutions $f \in C^2[0, \infty)$ with $df/d\hat{r}(0) = 0$ tend to zero when $r \rightarrow \infty$, the boundary condition (98) can be replaced by the following condition in the origin

$$\frac{df(0)}{d\hat{r}} = 0 \quad (101)$$

Therefore, the boundary problem represented by Eqs. (96), (97) and (98), corresponds to the following initial condition problem written in the Cauchy's normal form

$$\begin{aligned} \frac{df}{d\hat{r}} &= F \\ \frac{dF}{d\hat{r}} &= -5f - \left(\frac{1}{2} \sqrt{\frac{1-f}{2}} R + \frac{4}{\hat{r}} \right) F \end{aligned} \quad (102)$$

the initial condition of which is

$$f(0) = 1, \quad F(0) = 0 \quad (103)$$

References

ANTONIA R. A., SMALLEY R. J., ZHOU T., ANSELMET F., DANAILA L., Similarity solution of temperature structure functions in decaying homogeneous isotropic turbulence, *Phys. Rev. E*, **69**, 016305, 2004, DOI: 10.1103/PhysRevE.69.016305

- BAEV M. K., CHERNYKH G. G., On Corrsin equation closure, *Journal of Engineering Thermophysics*, **19**, pp. 154–169, no. 3, DOI: 10.1134/S1810232810030069
- BATCHELOR, G. K., Small-scale variation of convected quantities like temperature in turbulent fluid. Part 1. General discussion and the case of small conductivity, *Journal of Fluid Mechanics*, **5**, 1959, pp. 113–133
- BATCHELOR G. K., HOWELLS I. D., TOWNSEND A. A., Small-scale variation of convected quantities like temperature in turbulent fluid. Part 2. The case of large conductivity, *Journal of Fluid Mechanics*, **5**, 1959, pp. 134–139
- BATCHELOR G. K., *The Theory of Homogeneous Turbulence*. Cambridge University Press, Cambridge, 1953.
- CHASNOV, J., CANUTO V. M., ROGALLO R. S., Turbulence spectrum of strongly conductive temperature field in a rapidly stirred fluid. *Phys. Fluids A*, **1**, pp. 1698–1700, 1989, doi:10.1063/1.857535.
- CORRSIN S., The Decay of Isotropic Temperature Fluctuations in an Isotropic Turbulence, *Journal of Aeronautical Science*, **18**, pp. 417–423, no. 12, 1951.
- CORRSIN S., On the Spectrum of Isotropic Temperature Fluctuations in an Isotropic Turbulence, *Journal of Applied Physics*, **22**, pp. 469–473, no. 4, 1951., DOI: 10.1063/1.1699986.
- DE DIVITIIS N., Lyapunov Analysis for Fully developed Homogeneous Isotropic Turbulence, *Theoretical and Computational Fluid Dynamics*, DOI: 10.1007/s00162-010-0211-9.
- DE DIVITIIS N., Self-Similarity in Fully Developed Homogeneous Isotropic Turbulence Using the Lyapunov Analysis, *Theoretical and Computational Fluid Dynamics*, DOI: 10.1007/s00162-010-0213-7.
- DONZIS D. A., SREENIVASAN K. R., YEUNG P. K., The Batchelor Spectrum for Mixing of Passive Scalars in Isotropic Turbulence, *Flow, Turbulence and Combustion*, **85**, pp. 549–566, no. 3–4, DOI: 10.1007/s10494-010-9271-6

- GEORGE W. K., A theory for the self-preservation of temperature fluctuations in isotropic turbulence. Technical Report 117, Turbulence Research Laboratory, January 1988.
- GEORGE W. K., "Self-preservation of temperature fluctuations in isotropic turbulence," in *Studies in Turbulence*, Springer, Berlin, 1992.
- GIBSON, C. H., SCHWARZ W. H., The Universal Equilibrium Spectra of Turbulent Velocity and Scalar Fields, *Journal of Fluid Mechanics*, **16**, 1963, pp. 365–384
- GRANT, H.L., HUGHES, B.A., VOGEL, W.M., MOILLIET, A., Spectrum of temperature fluctuations in turbulent flow, *Journal of Fluid Mechanics*, **34**, 1968, pp. 423-442
- VON KÁRMÁN, T., HOWARTH, L., On the Statistical Theory of Isotropic Turbulence., *Proc. Roy. Soc. A*, **164**, 14, 192, 1938.
- VON KÁRMÁN, T., LIN, C. C., On the Concept of Similarity in the Theory of Isotropic Turbulence., *Reviews of Modern Physics*, **21**, 3, 516, 1949.
- LEHMANN, E. L., *Elements of Large-sample Theory*. Springer, 1999.
- MADOW, W. G., Limiting Distributions of Quadratic and Bilinear Forms., *The Annals of Mathematical Statistics*, Vol. 11, No. 2, (Jun. 1940), 125–146, 1940.
- MILLS, R. R. JR., KISTLER, A. L., O'BRIEN, V., CORRSIN, S., Turbulence and temperature fluctuations behind a heated grid, NACA-TN-4288, August 1958.
- MYDLARSKI, L., WARHAFT, Z., Passive scalar statistics in high-Péclet-number grid turbulence, *Journal of Fluid Mechanics*, **358**, 1998, pp. 135–175
- OAKEY, N. S., Determination of the rate of dissipation of turbulent energy from simultaneous temperature and velocity shear microstructure measurements, *J. Phys. Oceanogr.*, **12**, 1982, pp. 256-271
- OBUKHOV, A. M., The structure of the temperature field in a turbulent flow. *Dokl. Akad. Nauk.*, CCCP, **39**, 1949, pp. 391.

- OGURA, Y., Temperature Fluctuations in an Isotropic Turbulent Flow, *Journal of Meteorology*, **15**, 1958, pp. 539-546
- SREENIVASAN K. R., TAVOULARIS S., HENRY R., CORRSIN S., Temperature fluctuations and scales in grid-generated turbulence., *Journal of Fluid Mechanics*, **100**, 1980, pp. 597–621, doi:10.1017/S0022112080001309
- TRUESDELL, C., *A First Course in Rational Continuum Mechanics*, Academic, New York, 1977.
- VENTSEL, E. S., *Theorie des probabilites*. Ed. Mir, CCCP, Moskow, 1973.
- WARHAFT Z., LUMLEY J. L., An experimental study of the decay of temperature fluctuations in grid-generated turbulence. *Journal of Fluid Mechanics*, **88**, 1978, pp. 659–684, doi:10.1017/S0022112078002335

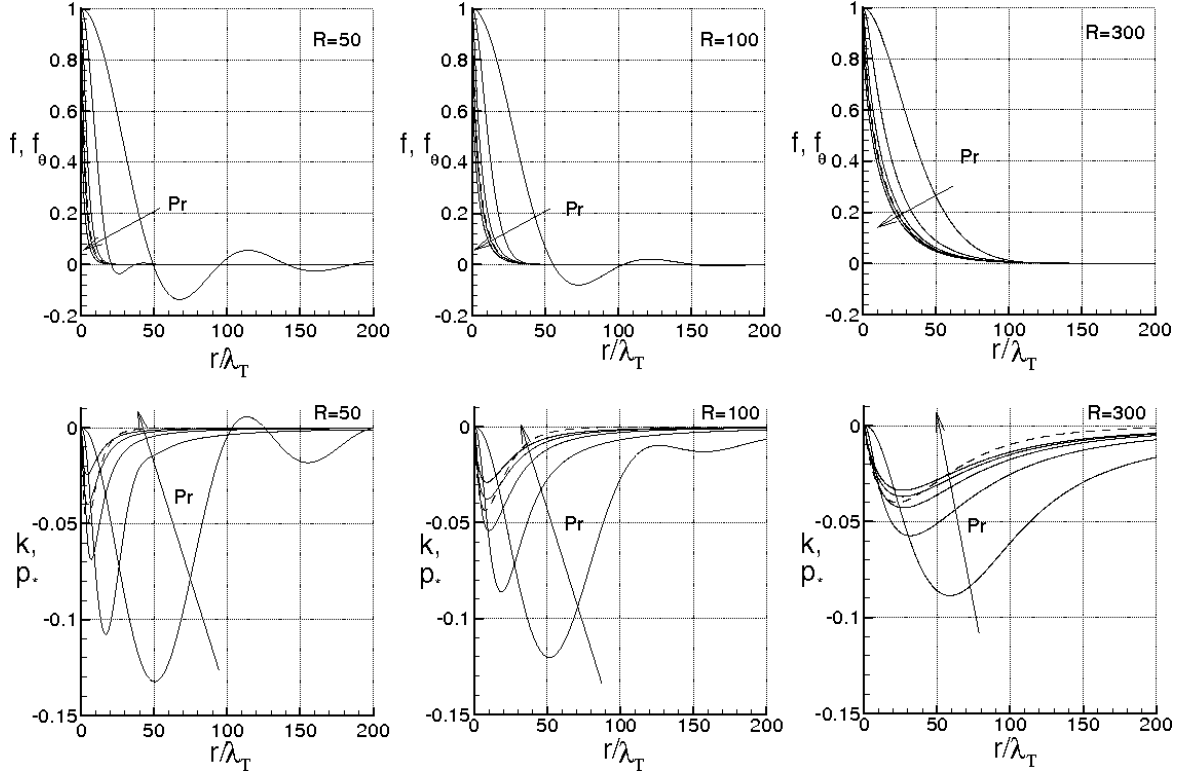


Figure 3: Correlation functions for $Pr=10^{-3}, 10^{-2}, 0.1, 1.0$ and 10 , at different Reynolds numbers. Top: velocity correlation f (dashed line) and temperature correlation f_θ (solid lines). Bottom: triple velocity correlation k (dashed line) and triple velocity-temperature correlation p_* (solid lines)

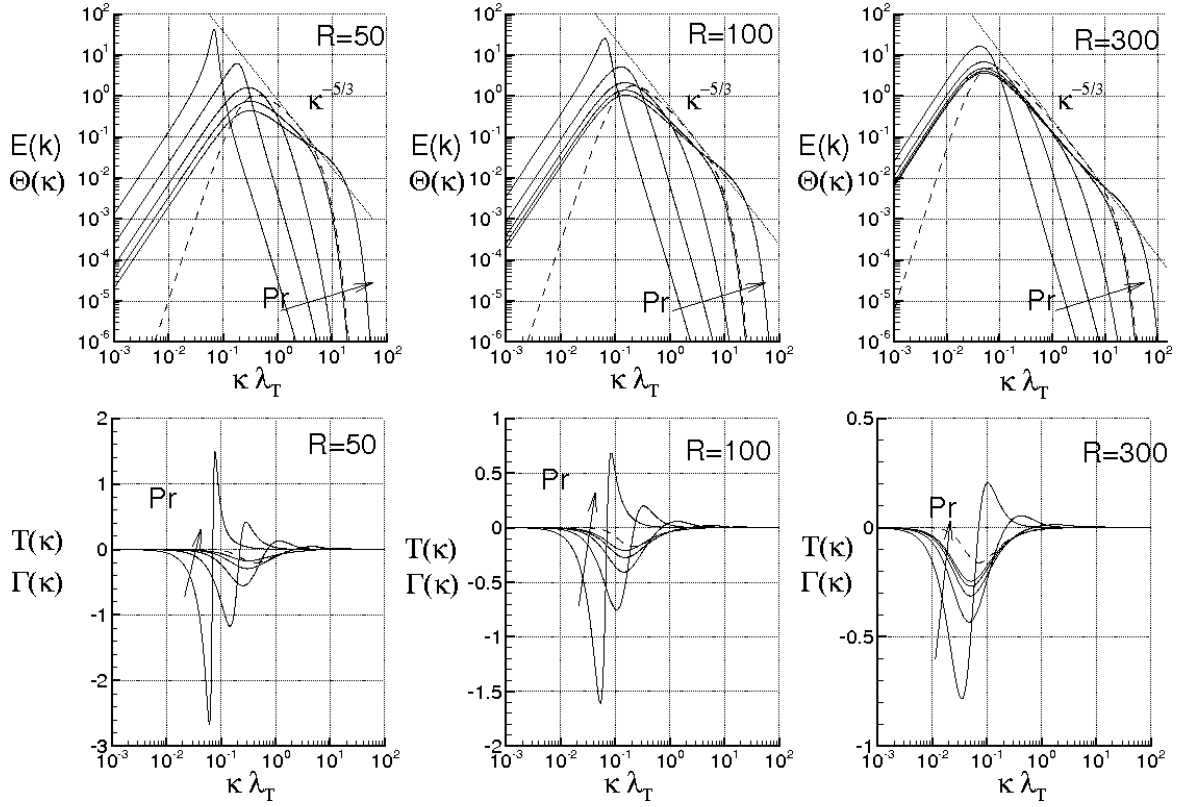


Figure 4: Spectra for $Pr = 10^{-3}, 10^{-2}, 0.1, 1.0$ and 10 , at different Reynolds numbers. Top: kinetic energy spectrum $E(\kappa)$ (dashed line) and temperature spectrum $\Theta(\kappa)$ (solid lines). Bottom: velocity transfer function $T(\kappa)$ (dashed line) and temperature transfer function $\Gamma(\kappa)$ (solid line)

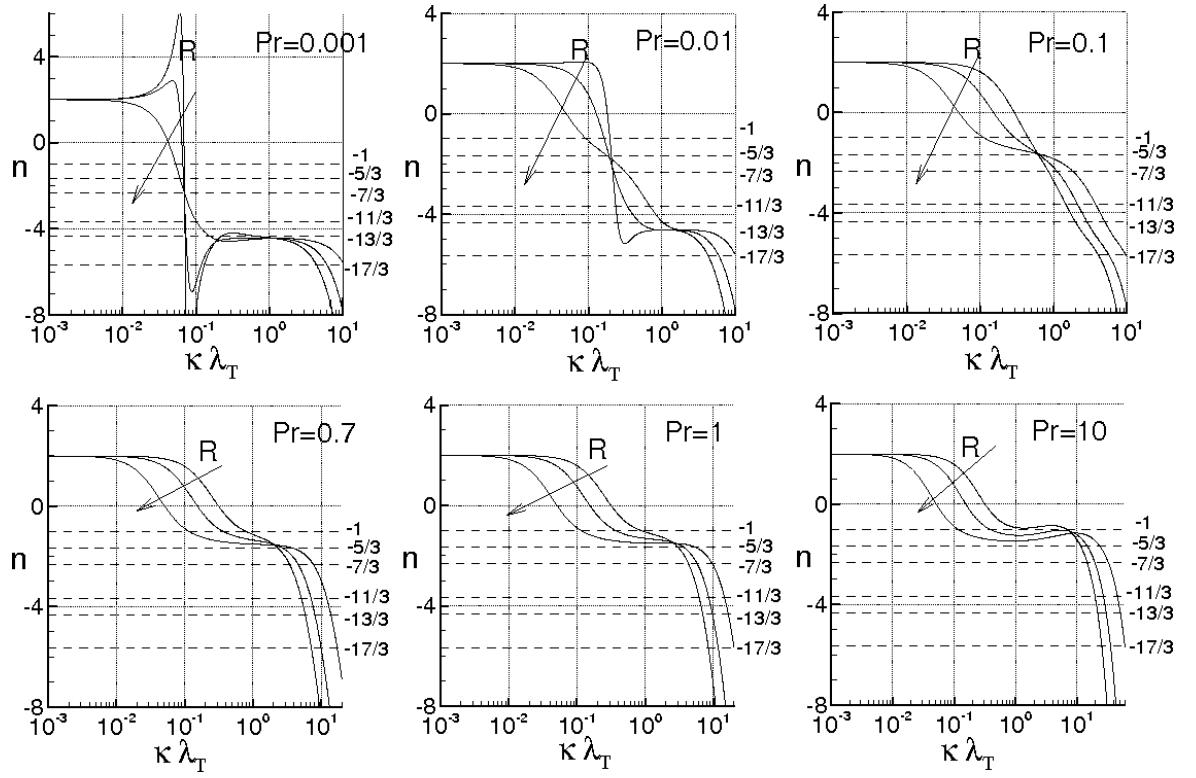


Figure 5: Scaling exponent of the temperature spectrum calculated for $Re = 50, 100$ and 300 , at different values of the Prandtl's number.

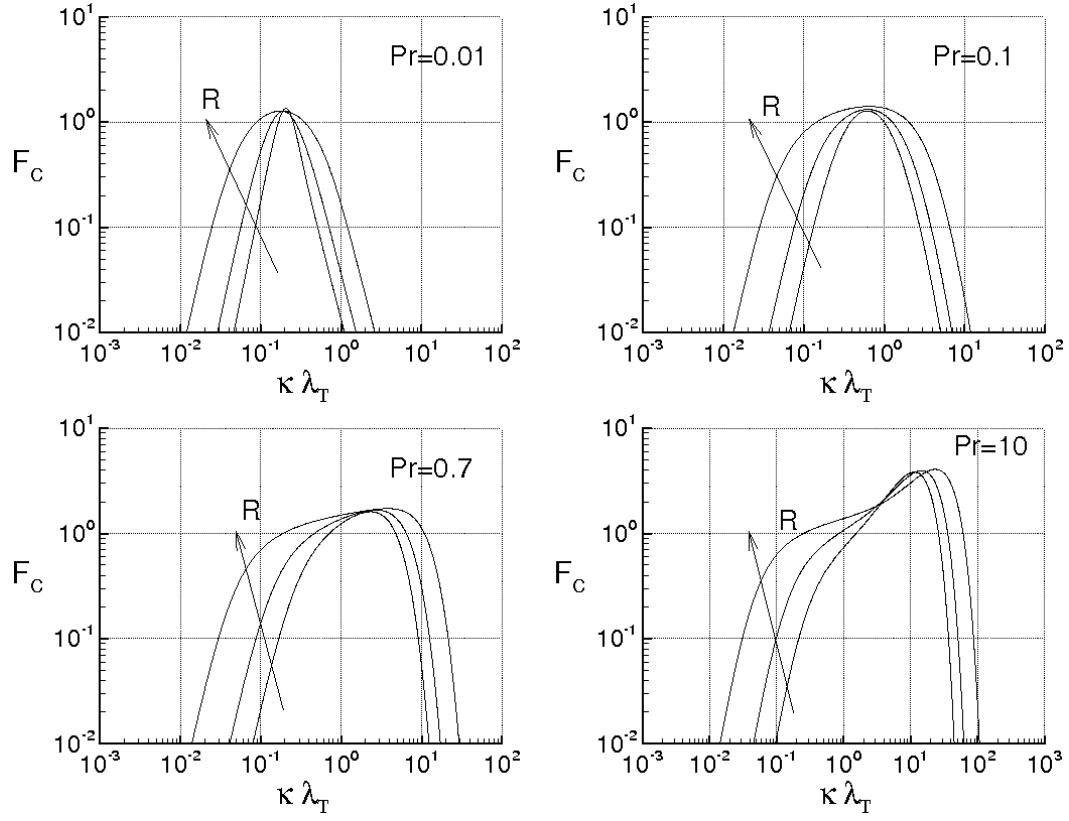


Figure 6: Corrsin function for $R=50, 100$ and 300 , at different values of Prandtl's numbers.

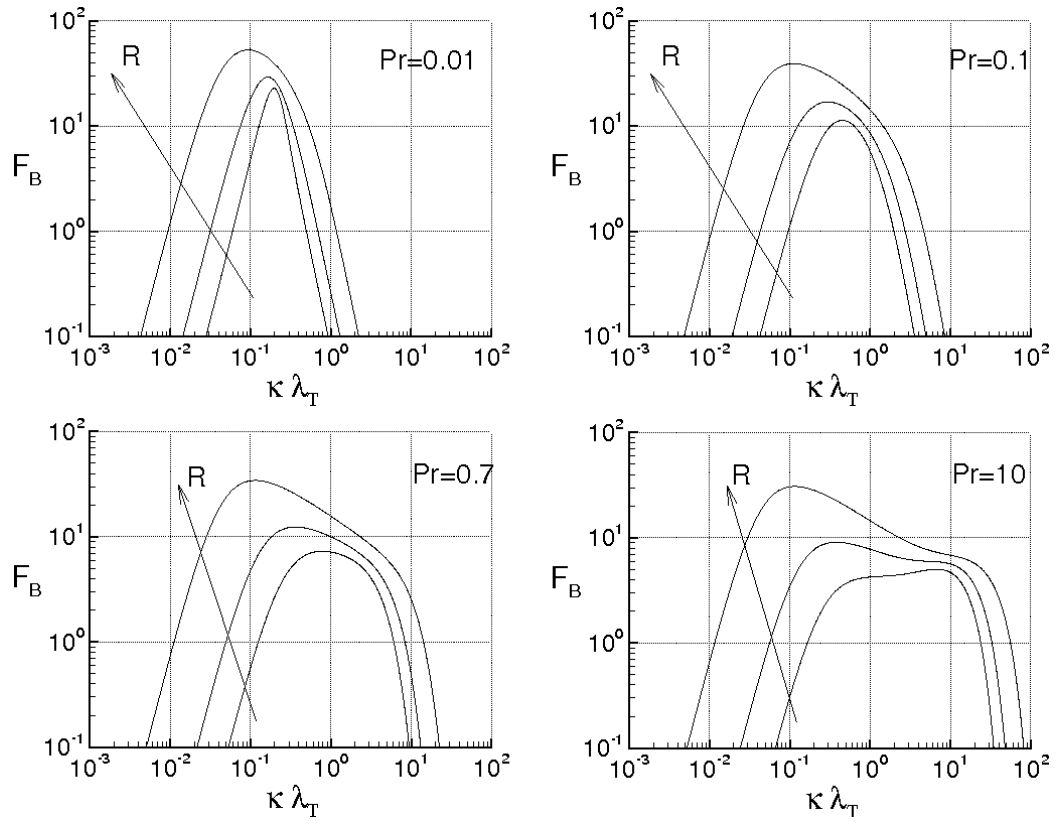


Figure 7: Batchelor's function for $R= 50, 100$ and 300 , at different values of Prandtl's numbers.

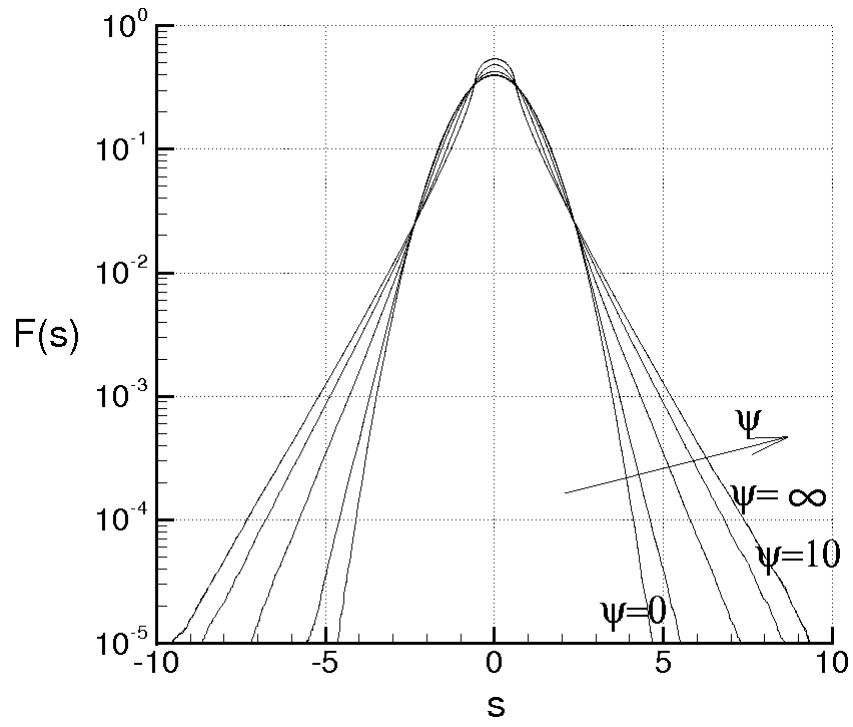


Figure 8: Distribution function of the longitudinal temperature derivatives, at different values of $\psi = C\sqrt{Pr} \bar{R}$

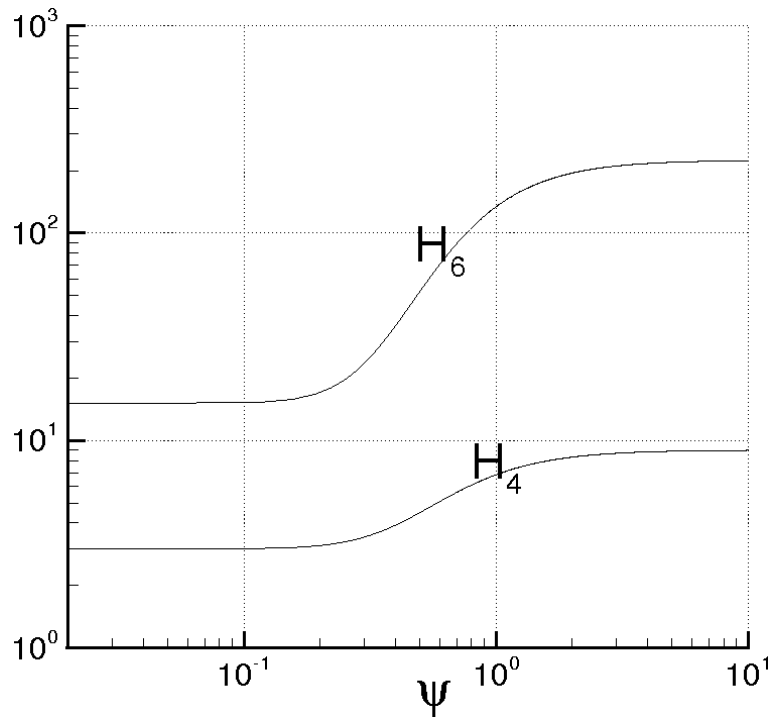


Figure 9: Dimensionless statistical moments, H_4 and H_6 of $\partial\vartheta/\partial r$ in function of the parameter ψ .